Solutions 7

Exercise 13.4

Since $\text{var}(s[n]) = a^{2n+2}\sigma_s^2 + \sigma_u^2 \sum_{k=0}^{n} a^{2k}$, we have

$$c_s[m,n] = a^{m-n}(a^{2n+2}\sigma_s^2 + \sigma_u^2 \sum_{k=0}^{n} a^{2k}) = a^{m-n}\text{var}(s[n]) = a^{m-n}c_s[n,n]$$

Exercise 13.6

From (13.14), we know that $\mathbb{E}(\mathbf{s}[n]) = \mathbf{A}^{n+1}\boldsymbol{\mu}_s$. Let the eigen-decomposition of $\mathbf{A}$ be $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T$. Thus, $\mathbf{A}^{n+1} = \mathbf{V}\boldsymbol{\Lambda}^{n+1}\mathbf{V}^T$

$$\mathbb{E}(\mathbf{s}[n]) = \mathbf{V}\boldsymbol{\Lambda}^{n+1}\mathbf{V}^T\boldsymbol{\mu}_s = \sum_{i=1}^{P} a_i \lambda_i^{n+1}\mathbf{v}_i$$

where $a_i = (\mathbf{V}^T\boldsymbol{\mu}_s)_i$ and $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_P)$. If any $|\lambda_i| > 1$, $\mathbb{E}(\mathbf{s}[n]) \to \infty$ and $|\lambda_i| < 1$, $\mathbb{E}(\mathbf{s}[n]) \to 0$.

Exercise 13.10

From (13.38) to (13.42) with $a = 1$ and $\sigma_u^2 = 0$, so that $s[n] = s[n-1] = A$. We can get the prediction stage

$$\hat{A}[n|n-1] = \hat{A}[n-1|n-1] \qquad M[n|n-1] = M[n-1|n-1]$$

$$K[n] = \frac{M[n-1|n-1]}{\sigma^2 + M[n-1|n-1]}$$

$$\hat{A}[n|n] = \hat{A}[n-1|n-1] + K[n](x[n] - \hat{A}[n-1|n-1])$$

$$M[n|n] = (1 - K[n])M[n-1|n-1]$$

or changing the notation, we have

$$K[n] = \frac{M[n-1]}{\sigma^2 + M[n-1]}$$

$$\hat{A}[n] = \hat{A}[n-1] + K[n](x[n] - \hat{A}[n-1])$$

$$M[n|n] = (1 - K[n])M[n-1]$$

These equations are just from (12.34) to (12.36). Hence, from section 12.6, we have

$$\hat{A}[n] = \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2/n+1} \cdot \frac{1}{n+1} \sum_{k=0}^{n} x[k]$$

$$M[n-1] = \frac{\sigma_A^2 \sigma^2}{n\sigma_A^2 + \sigma^2}$$

$$K[n] = \frac{\sigma_A^2}{(n+1)\sigma_A^2 + \sigma^2}$$

Exercise 13.11

From (13.39), (13.40) and (13.42)

$$M[n|n-1] = 0.81M[n-1|n-1] + 1$$

$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

$$M[n|n] = (1 - K[n])M[n|n-1]$$

where $M[-1|-1] = \sigma_s^2 = 1$

For (a), the gain $\to 1$ since the observations become less noisy and thus $\hat{s}[n|n] \to x[n]$, but (c) is opposite. For (b) the gain and MMSE sequences attain a steady state after a few iterations. Figures are omitted.

Exercise 13.13

Since $\mathbb{E}(s[n])$ is known, the MMSE estimation of $s^{'}[n] = s[n] - \mathbb{E}(s[n])$ is $\hat{s}^{'}[n] = \hat{s}[n] - \mathbb{E}(s[n])$ and the MMSE is the same as for $\hat{s}[n]$. Thus, $M[n|n]$ and $M[n|n-1]$ do not change. Also, then $K[n]$ does not change.

Prediction:

$$\hat{s}^{'}[n|n-1] = a\hat{s}^{'}[n-1|n-1]$$

$$\hat{s}[n|n-1] - \mathbb{E}(s[n]) = a(\hat{s}[n-1|n-1] - \mathbb{E}(s[n-1]))$$

Since $\mathbb{E}(s[n]) = a\mathbb{E}(s[n-1])$, we have $\hat{s}[n|n-1] = a\hat{s}[n-1|n-1]$

Correction:

$$\hat{s}^{'}[n|n] = \hat{s}^{'}[n|n-1] + K[n](x^{'}[n] - \hat{s}^{'}[n|n-1])$$

$$\hat{s}[n|n] - \mathbb{E}(s[n]) = \hat{s}[n|n-1] - \mathbb{E}(s[n]) + K[n](x[n] - \mathbb{E}(x[n]) - \hat{s}[n|n-1] + \mathbb{E}(s[n]))$$

But $\mathbb{E}(x[n]) = \mathbb{E}(s[n]) + \mathbb{E}(w[n]) = \mathbb{E}(s[n])$. Thus, we have the same equations as before. The only differences arises in the initialization since $\mu_s \neq 0$

Exercise 13.14

In steady state, we have $M[n|n] = M[\infty]$, $M[n|n-1] = M_p[\infty]$ and from (13.42)

$$M[\infty] = (1 - K[\infty])M_p[\infty] < M_p[\infty]$$

since $K[\infty] < 1$. Thus, for large n, $M[n|n-1] > M[n-1|n-1]$. This is reasonable since $s[n]$ is harder to estimate than $s[n-1]$ based on $\{x[0], x[1], ..., x[n-1]\}$ due to the added variability (not sure) of the $u[n]$ noise term.

Exercise 13.15

From (13.38), $\hat{s}[n+1|n] = a\hat{s}[n|n]$. Now if $\sigma^2_{n+1} \to \infty$, the future measurements will be useless so that the corrected estimation will be prediction or will be based on only $\{x[0], ..., x[n]\}$. Thus,

$$\hat{s} \to \hat{s}[n+1|n] = a\hat{s}[n|n]$$

$$\hat{s}[n+2|n+1] = a\hat{s}[n+1|n+1] = a^2\hat{s}[n|n]$$

etc.